Nonstationary Pomeau-Manneville intermittency in systems with a periodic parameter change

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Pomeau-Manneville intermittency in nonstationary systems is investigated. If one of the parameters characterizing a dynamical system is changed periodically, periodic orbits may appear even when the value of this parameter remains in a range which, in the stationary case, yields chaotic behavior. This property may be used for the control of systems exhibiting intermittency. If the parameter change is not large enough, a periodic orbit does not appear but the distribution of the laminar phases is modified. In the case of type I intermittency, this means a broadening of such a distribution or, alternatively, a splitting of its right peak. We present a theory of these phenomena. Numerical simulations both for one-dimensional maps and for flows support our predictions. Some of the phenomena discussed here were observed earlier in time series of heart rate variability.

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I. INTRODUCTION

In nonlinear dynamics the term "intermittency" means a switching of the state of the system between two different types of behavior, e.g., a periodic and a chaotic behavior. The phenomenon was studied by Pomeau and Manneville who, at the end of the 1970s, were the first to investigate intermittency in the temporal domain [1,2]. They also classified the elementary kinds of intermittency phenomena grouping them into three categories according to the type of bifurcation that is associated with the given intermittency. Today the best known is type I intermittency which is the effect of the proximity of the system to a tangent bifurcation (i.e., a saddle-node bifurcation). In this kind of bifurcation two solutions appear—a hyperbolic saddle and a node. It is in this range of control parameter space just before the tangent bifurcation occurs that one observes a chaotic motion consisting of laminar phases (in which the motion is almost periodic) and of chaotic bursts.

Type I intermittency has been observed in many experiments, and has also been theoretically investigated [3]. The authors of Ref. [3] showed that there is a recursion relation valid close to the bifurcation point:

$$y_{n+1} = y_n + \alpha y_n^2 + \varepsilon \tag{1}$$

where α is a parameter depending on the shape of the one-dimensional map, and ε is proportional to the difference between the actual and the critical values of the control parameter. The variable y_n is a measure of the distance from the bifurcation point along the line $y_{n+1}=y_n$. If $\varepsilon<0$ Eq. (1) has two fixed points—one stable and one unstable. If $\varepsilon=0$, the plot of Eq. (1) becomes tangent to the diagonal and the bifurcation occurs. If $\varepsilon>0$, there is no real solution of Eq. (1) but we obtain the laminar phase instead: successive iterates of Eq. (1) differ only by a small value. For small ε the number of iterates within the laminar phase is large. Note that Eq. (1) is a model of the laminar phase only so that the chaotic bursts must be introduced into simulations artificially.

Using this simple model, Hirsch showed many properties of type I intermittency. For example, he obtained the maximal length of laminar phases:

$$l_{\max} = \frac{2}{\sqrt{\alpha\varepsilon}} \arctan\left(\frac{y_0}{\sqrt{\varepsilon/\alpha}}\right) \tag{2}$$

and the probability distribution for the lengths l of laminar phases:

$$P(l) = \frac{\varepsilon}{2y_0}\left\{1 + \tan^2\left[\arctan\left(\frac{y_0}{\sqrt{\varepsilon/\alpha}}\right) - l\sqrt{\varepsilon\alpha}\right]\right\} \tag{3}$$

where ε is proportional to the difference between the actual and the critical values of the control parameter, α is the second derivative of the map (1), and y_0 defines the end of the intermittency channel.

Pomeau and Manneville also defined two other kinds of intermittency: type II intermittency, related to the Hopf bifurcation, and type III, occurring in systems in which reverse period doubling bifurcation is possible [2].

Intermittency in stationary states has been the subject of a very large number of papers. But in real complex systems, one can seldom observe ideal stationary states. In many of these systems we should take into account random noise, which can modify the behavior of a system in a major way. Another effect may be due to a nonrandom, periodic or not, control parameter change. In their work, Hirsch *et al.* [3] investigated the effect of additive Gaussian noise on the properties of type I intermittent system. A system with a periodic parameter change or subject to parametric noise was investigated numerically in Ref. [4]. Intermittency in such a nonstationary system is quite different from that described earlier by other authors. Depending on the scheme of parameter change, either a modification of the probability distribution of the lengths of laminar phases is obtained or, in some cases, intermittency can be destroyed and a periodic orbit appears [4]. As we will see below, this last result can be very useful in controlling chaos in systems with Pomeau-Manneville intermittency.

Control of chaos is one of the most important applications of nonlinear dynamics [2,5,6]. In many cases, it is desirable to force a system in a chaotic state to maintain periodic behavior. There are several methods of controlling chaotic dynamical systems [2] in which small changes of the accessible control parameter are applied.

The aim of the best known method from this group [the Ott-Grebogi-Yorke (OGY) algorithm [5]] is to stabilize an unstable periodic orbit which belongs to the set forming the chaotic attractor. When the trajectory of the system comes close to the desired unstable periodic orbit, small changes of control parameter are applied to force the trajectory onto the stable manifold. This assures that the trajectory will not escape from the vicinity of the periodic orbit. The OGY method has been successfully applied to various physical [7], chemical [8,9], and medical [10] systems.

To make possible the control of the dynamics of some more complex systems, several methods similar to the OGY algorithm were developed. Triandaf and Schwarz introduced the idea of tracking unstable orbits [6]. The so-called continuation method allows one to follow unstable orbits over a large range of parameter values and through the period-doubling bifurcations [11]. Tracking allows one not only to stabilize an unstable orbit embedded in a chaotic attractor but also to force a system to remain in the vicinity of an unstable orbit when some stable orbit is present simultaneously (e.g., beyond the point of a period-doubling bifurcation). Orbit tracking was successfully implemented in controlling chaos in such systems as a modulated CO_2 laser [12]. It should be emphasized that this method can be applied also for nonstationary systems, i.e., when the control parameter changes with the time [13].

The method of controlling on-off intermittent dynamics proposed in [14] is also based on the OGY method. The main idea of this method is to not allow a trajectory of the system to escape from the vicinity of an "off" state. For this purpose, parameter changes are computed similarly as in the original OGY method.

We present a theory of intermittency in discrete dynamical systems and flows. In Sec. II, we obtain analytically the laminar phase length distribution which occurs in type I intermittency in a system with a dichotomous control parameter change. In Sec. III, we describe the appearance of periodic orbits due to a periodic parameter change in a system exhibiting a Pomeau-Manneville intermittency. This effect may be used to control the dynamics of such a system. All the theoretical considerations are supported by numerical experiments in discrete systems and flows. In Sec. IV we present the conclusions.

II. EFFECT OF DICHOTOMOUS PARAMETER CHANGE ON THE PROBABILITY DISTRIBUTION OF LAMINAR PHASE LENGTHS IN TYPE I INTERMITTENCY

Consider type I intermittency in a system in which the value of the control parameter is changed periodically. As will be shown in Sec. III, if the parameter change exceeds a certain threshold, a stable periodic orbit may appear. But if the changes of the control parameter are too small, one will not obtain such an orbit. The trajectory of system will remain in the intermittency state. One can expect that such an intermittency differs from the one in the stationary state.

This case was investigated numerically for the logistic map in Ref. [4] and it was found that, if a stable orbit does not appear, two characteristic phenomena can be observed: (a) a broadening of the laminar phase length distribution and (b) a splitting of the right peak of this distribution. The authors used the same value of control parameter change every time and observed that the final effect depends on the frequency of the parameter change. The main motivation for this research was that similar effects can be observed in real systems. In Refs. [4,15] laminar phase length probability density distributions characteristic for type I intermittency were found for 24-h heart rate variability recordings The splitting of the right peak was clearly seen for these distributions (Figs. 1 and 2).

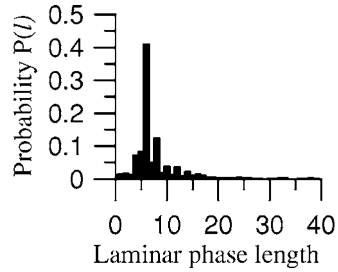

FIG. 1. Laminar phase probability density distribution for the 24 h heart rate variability recording of the patient DWD (after [15]). This was a postmyocardial infarct patient with many ventricular arrhythmia events during the recording and a cardiac arrest 1 h before its end.

In this section, we form a theory that explains the results of the numerical experiments of Ref. [4]. Let us consider a system in which a sequence of iterations is repeated: n iterations using a specific base value of the control parameter are followed by m iterations with this parameter changed to a different value (n and m are natural numbers). We will denote by $G(\Delta a;x_i)$ the $(n+m)T$th iteration of the map and assume that for the first n iterations the control parameter value is a while for the next m iterations it is $a+\Delta a$. The sequence of control parameter change is repeated T times. Next, the whole procedure is repeated. Here, the notation $G(0;x_i)$ means simply the $(n+m)T$th iteration of the stationary map.

When our system is within the parameter range where intermittency appears, we can write

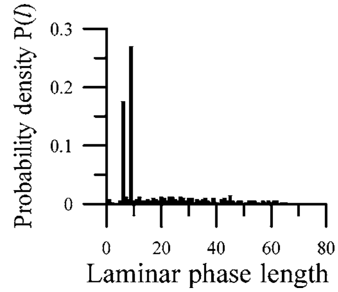

FIG. 2. Laminar phase probability density distribution for the 24 h heart rate variability recording of the patient FTCH (after [4]). The patient had sinus and atrial heart rhythm with an extremely increased number of ventricular arrhythmia (around 50% of heart cycles were ectopic).

$$G(0;x_i) \neq x_i$$

where x_i belongs to the period-T orbit.

If we were in the range of periodic motion, we would have of course

$$G(0;x_i) = x_i, \quad i = 1,2, \ldots ,T. \tag{4}$$

Let us define

$$\varepsilon_i = \frac{G(0;x_i) - x_i}{n+m}. \tag{5}$$

The value of ε_i is a measure of the minimal distance between the plot of the Tth iteration of the map and the diagonal. Note that ε_i given by Eq. (5) is a different parameter from the ε present in Eq. (1) but it is proportional to it. Thus, the maximal laminar phase length l depends on the value of ε_i [2,3]:

$$l \propto |\varepsilon_i|^{-1/2}. \tag{6}$$

The modulus in Eq. (6) comes from the fact that ε_i defined by Eq. (5) can be both positive and negative.

Now we define a similar value for the nonstationary case:

$$\varepsilon_i' = \frac{G(\Delta a;x_i) - x_i}{n+m}. \tag{7}$$

We assume that the new value of maximal laminar phase length (and also the position of the right peak of the distribution) satisfies the dependence

$$l_i' \propto |\varepsilon'_i|^{-1/2}, \tag{8}$$

where the index i means that we consider a trajectory which would be in the vicinity of the point x_i in the first iteration using the base value of the control parameter. If we know the laminar length l in the stationary case, we can compute the nonstationary value as

$$\frac{l'_i}{l} = \sqrt{\frac{\varepsilon_i}{\varepsilon'_i}} = \sqrt{\frac{(n+m)\varepsilon_i}{\frac{\partial G(0;x_i)}{\partial \Delta a}\Delta a + (n+m)\varepsilon_i}} \tag{9a}$$

where the left side of Eq. (7) was expanded into a series with respect to Δa and all nonlinear terms were neglected. As we will show in the next sections, this approximation works very well.

All the difficulty in predicting the position of the right peak of the distribution is to compute the derivative of the function G. This can be done if we treat G as a composition of the map $f(a;x)$. This leads to the equation

$$\frac{\partial G(0;x_i)}{\partial \Delta a} = \sum_{q=1}^{T} \sum_{j=qn+(q-1)m}^{q(n+m)-1} \left(\prod_{k=j+1}^{T(n+m)} \frac{\partial}{\partial x} f(a;x_{i+k-1}) \right) \times \frac{\partial}{\partial a} f(a;x_{i+j-1}). \tag{10a}$$

For a derivation of Eq. (10a) see the Appendix.

The above described method of finding the position of the right peak of the distribution can be simply generalized to the case in which the control parameter is altered between more than two values. In such a case, one of the values of the parameter change Δa has to be set as the base value and other values of the parameter change are determined by this base value. The new position of the right peaks can be computed again from an equation very similar to Eq. (9a) but the derivative of the function G has to be computed from a slightly more complicated equation than Eq. (10a).

Suppose that the first n iterations run without parameter change, the next m_1 iterations with the parameter change equal to Δa, and the next m_2 with Δa_2. We have k possible values in all (some of them may be equal). Let us define

$$\alpha_p = \frac{\Delta a_k}{\Delta a} \quad (p = 1, \ldots ,k \text{ and } \alpha_1 = 1),$$

$$M_0 = m_0 \equiv n,$$

$$n + \sum_{p=1}^{j} m_p = M_j,$$

$$N \equiv M_k = n + \sum_{p=1}^{k} m_p.$$

Now we can write the derivative of G instead of Eq. (10a) as

$$\frac{\partial G(0;x_i)}{\partial \Delta a} = \sum_{q=1}^{T} \sum_{p=1}^{k} \sum_{j=(q-1)N+M_{p-1}}^{(q-1)N+M_p-1} \left(\prod_{l=j+1}^{TN} \frac{\partial}{\partial x} f(a;x_{i+l-1}) \right) \times \alpha_p \frac{\partial}{\partial a} f(a;x_{i+j-1}), \tag{10a$'$}$$

and, instead of Eq. (9), we have

$$\frac{l'_i}{l} = \sqrt{\frac{\varepsilon_i}{\varepsilon'_i}} = \sqrt{\frac{N\varepsilon_i}{\frac{\partial G(0;x_i)}{\partial \Delta a}\Delta a + N\varepsilon_i}}. \tag{9a$'$}$$

When T is the common denominator of $(n+m)$ and N, instead of the function G, we can define more simply the function G':

$$G(\Delta a;x_i) = TG'(\Delta a;x_i),$$

$$\frac{\partial G'(0;x_i)}{\partial \Delta a} = \sum_{j=n}^{n+m-1} \left(\prod_{k=j+1}^{n+m} \frac{\partial}{\partial x} f(a;x_{i+k-1}) \right) \frac{\partial}{\partial a} f(a;x_{i+j-1}). \tag{10b}$$

This equation contains a smaller number of sums than Eq. (10a). Instead of Eq. (9a) we obtain

$$\frac{l'_i}{l} = \sqrt{\frac{\varepsilon_i}{\varepsilon'_i}} = \sqrt{\frac{(n+m)\varepsilon_i}{T\frac{\partial G'(0;x_i)}{\partial \Delta a}\Delta a + (n+m)\varepsilon_i}}. \tag{9b}$$

When there are more than two different values of the control parameter, we use, instead of Eqs. (9a′) and (10a′)

$$\frac{l'_i}{l} = \sqrt{\frac{\varepsilon_i}{\varepsilon'_i}} = \sqrt{\frac{N\varepsilon_i}{T\frac{\partial G'(0;x_i)}{\partial \Delta a}\Delta a + N\varepsilon_i}}, \qquad (9b')$$

$$\frac{\partial G'(0;x_i)}{\partial \Delta a} = \sum_{p=1}^{k}\sum_{j=M_{p-1}}^{M_p-1}\left(\prod_{l=j+1}^{N}\frac{\partial}{\partial x}f(a;x_{i+l-1})\right)\alpha_p\frac{\partial}{\partial a}f(a;x_{i+j-1}). \qquad (10b')$$

A. Example: The logistic map

Consider the well-known logistic map:

$$x_{n+1} = ax_n(1-x_n). \qquad (11)$$

The widest periodic window for the logistic map appears at the value of the control parameter $a=a_C=1+\sqrt{8} \cong 3.828\,427\,125\ldots$, at which a saddle-node bifurcation occurs [3]. If the value of a is slightly less than a_C, our system exhibits type I intermittency. The periodic orbit within this window appears as an effect of the tangent bifurcation and consists of three points: $x_1=0.159\,928\,815\ldots$, $x_2=0.514\,355\,268\ldots$, and $x_3=0.956\,317\,843\ldots$.

Let us fix the base value of a in Eq. (11) equal to 3.828. The values of ε computed from Eq. (5) are equal: $\varepsilon_1=3.333\times10^{-4}$, $\varepsilon_2=8.660\times10^{-4}$, and $\varepsilon_3=-9.532\times10^{-5}$.

Now let us consider increasing the value of a to 3.828 427 122 5 at every second iteration. Using Eqs. (9a) and (10a) we can compute the new position of the right peak for every point belonging to the orbit. We obtain

$$\frac{l'_1}{l} \cong 1.413\,93, \quad \frac{l'_2}{l} \cong 1.414\,71, \quad \frac{l'_3}{l} \cong 1.413\,92.$$

These three values are almost identical. Since the laminar phase length is an integer, in this case the three peaks cannot be recognized as long as the maximum of laminar phase length in the stationary case is less than 10 000. For the base value of the control parameter we had assumed, this length is close to 40 so we are not able to observe the splitting of the right peak of the distribution. Instead a broadening of the distribution will be observed. Figure 3 depicts the stationary (a) and the nonstationary (b) distributions. Computing the ratio of the positions of the maxima of the distributions yields $76/54=1.4074$—in a very good agreement with the theoretical prediction. The small difference between the numerical result and that obtained theoretically is due to the approximations used and to the fact that the observed laminar phase length can only be a natural number.

Now let us consider $n=10$ iterations with $a=3.828$ and $m=2$ (at which the value of a is increased to 3.828 427 122 5). The ratio of the maxima of the laminar phase lengths is equal to about 1.167 for $i=1$ or 3 and 0.960 for $i=2$. These values are quite different. This means that a splitting of the right peak of the distribution will be visible. Figure 4 shows such a nonstationary distribution with two right peaks: the ratio l'/l is now $63/54=1.167$ and $52/54=0.963$, respectively. Numerous examples of type I intermittency with a distribution of laminar phases and a split right peak similar to that in Fig. 4 were measured in human heart rate variability [4,15] (Figs. 1 and 2).

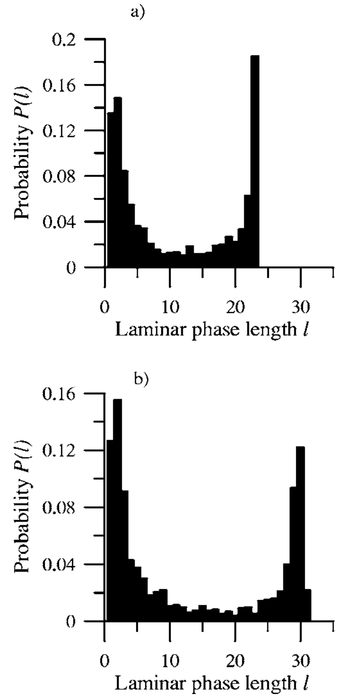

FIG. 3. Probability distribution of the lengths of laminar phases for the logistic map for the stationary case (a) with control parameter set equal to 3.828, and for the nonstationary case when the control parameter is alternated between 3.828 and 3.828 427 122 5 (b).

Now consider that we have three, instead of two, different values of the control parameter in successive iterations of the logistic map. For example, let us consider the first iteration with $a=3.828$, the second iteration with $a=3.8282$, and the third with $a=3.8279$. This leads to the following values of the parameters:

$$m_0 = n = 1, \quad m_1 = m_2 = 1, \quad \Delta a = 0.0002,$$

$$\alpha_1 = 1, \quad \alpha_2 = -0.5, \quad k = 2,$$

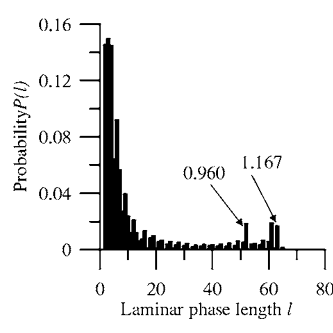

FIG. 4. Probability distribution of the lengths of laminar phases for the logistic map for the nonstationary case with $n=10$ and $m=2$. The arrows mark the ratio of the position of the given peak to the right peak of the distribution in the stationary case [Fig. 3(a)] in excellent agreement with the theoretical predictions.

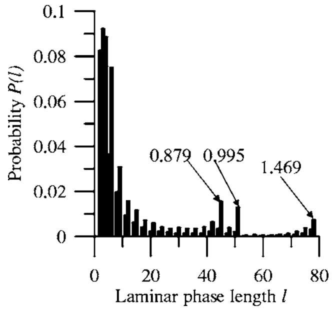

FIG. 5. Probability distribution of the lengths of laminar phases for the logistic map for the nonstationary case: three, instead of two, different values of the control parameter in successive iterations of the logistic map. The arrows mark the ratio of the position of the given peak to the right peak of the distribution in the stationary case [Fig. 3(a)] in excellent agreement with the theoretical predictions.

$$M_0 = 1, \quad M_1 = 2, \quad M_2 = N = 3.$$

Inserting these into Eq. (10b′) leads to

$$\frac{\partial G'(0;x_i)}{\partial \Delta a} = -0.5x_{i-1}(1-x_{i-1}) + a(1-2x_{i-1})x_{i-2}(1-x_{i-2}). \tag{12}$$

This equation, together with (9b′) allows us to compute the position of the right peaks of the distribution:

$$\frac{l'_1}{l} = 1.469, \quad \frac{l'_2}{l} = 0.995, \quad \frac{l'_3}{l} = 0.879.$$

The values obtained are completely different, so we expect in this case that the right peak of the probability distribution of the laminar phase lengths will split into three, not into two peaks. Figure 5 depicts such a distribution with the right peak split into three and the corresponding values of the ratios obtained numerically. Such a triple splitting of the right peak was also obtained for some of the cases of type I intermittency obtained for the heart rate variability [4,15].

B. Example: The Rössler system (a flow)

We will apply our theory to the well-known Rössler system [16], described by the three equations

$$\frac{dX}{dt} = -Y - Z,$$

$$\frac{dY}{dt} = X + AY,$$

$$\frac{dZ}{dt} = B + Z(X + C). \tag{13}$$

This system is described by three real variables and has three parameters A, B, and C. Let us set the values of the parameters $B=2.0$ and $C=4.0$. A will be our control parameter. The Rössler system then goes through a saddle-node bifurcation at $A=0.458$. If the value of A is slightly smaller than that, type I intermittency is obtained.

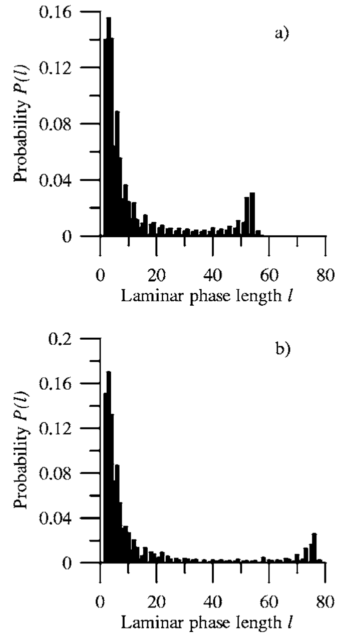

FIG. 6. Broadening of the distribution of the lengths of laminar phases for the Rössler system as a result of nonstationarity: (a) stationary case; (b) nonstationary case. For details see text.

To apply our theory developed for one-dimensional maps to this system, one should extract such a map. We formed the first return map of the successive maxima of Y. Figure 5 depicts a plot of this function. The third iteration of this function is shown in Fig. 6 (dots) together with the polynomial fit (solid line). We can clearly see the characteristic three narrow intermittency channels.

Set the values of the parameters $B=2.0$ and $C=4.0$. The base value of A will be set at 0.4576 and increased to 0.4581 for one time interval per three between the successive maxima of the Y variable. We obtain a broadening of the probability distribution. Such a distribution, compared to the stationary case [Fig. 6(a)] is shown in Fig. 6(b).

Extracting the map as above from Eq. (9a), one obtains that the function $(l/l')^2 = f(\Delta A)$ is linear. In fact, we can transform this equation into

$$\left(\frac{l}{l'}\right)^2 = \frac{\dfrac{\partial G(x;0)}{\partial \Delta A}}{(n+m)\varepsilon}\Delta A + 1. \tag{14}$$

The plot of this function is shown in Fig. 7 for several values of ΔA (squares). A linear function with a unit direction coefficient is an excellent approximation of this dependence (solid line).

Now, let us consider new values for the parameters: $A=0.45$, $B=3$, while C is changed between the two values 4.5705 and 4.570 55. In this case, we obtain numerically a characteristic splitting of the right peak into two.

A theoretical computation of the positions of split peaks is more difficult than in the case of the logistic map (or any system with discrete time), although we can approximate the functional dependence of the maximum of Y as a function of the previous maximum $f(A;x)$ on x. However, we do not

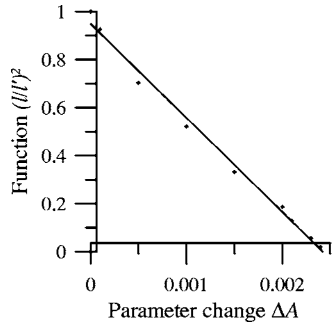

FIG. 7. The square of the ratio of the laminar phase length l' for the nonstationary Rössler system to the laminar phase length l in the stationary case is a linear function (fit, solid line) of the magnitude of the control parameter change ΔA (squares).

know the exact form of the dependence of this function on A. To solve this problem, we find a polynomial approximation of the function $f(A;x)$ for a few different values of A and obtain the required derivative $\partial f(A;x)/\partial A$ by numerical differentiation. To enhance the accuracy of this calculation, it is better to fit the polynomial only in the relatively small neighborhood of a maximum (each maximum may give a slightly different form of the approximating polynomial). We then compute the new positions of the right peaks and obtain

$$\frac{l'_1}{l} = 1.1699, \quad \frac{l'_2}{l} = 0.991, \quad \frac{l'_3}{l} = 0.979.$$

These three values do not differ very much but, since we are very close to the bifurcation point, the typical laminar phase length is much greater than 50. In such a case, differences of the order of 0.2 in the ratio let us discern only two peaks of the distribution (the second and third peaks are indistinguishable). This is clearly seen in Fig. 8.

Finally, one should stress that if the control parameter were changed independently of the value of Y (e.g., if there were a metronomic process forcing such changes), the function $f(A;x)$ would contain an infinite number of different values of the control parameter. We would then expect the distribution to be similar as in the case of parametric noise [15], i.e., there will be a long tail of the distribution extending toward large laminar phase lengths. Such a plot is shown on Fig. 9.

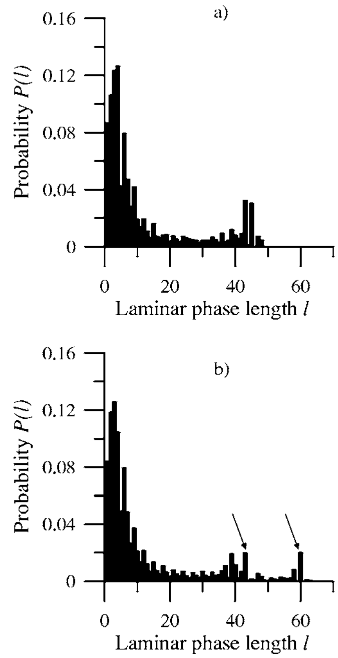

FIG. 8. Split peaks in the distribution of the laminar phases lengths for the Rössler system as a result of nonstationarity: (a) stationary case; (b) nonstationary case. For details see text.

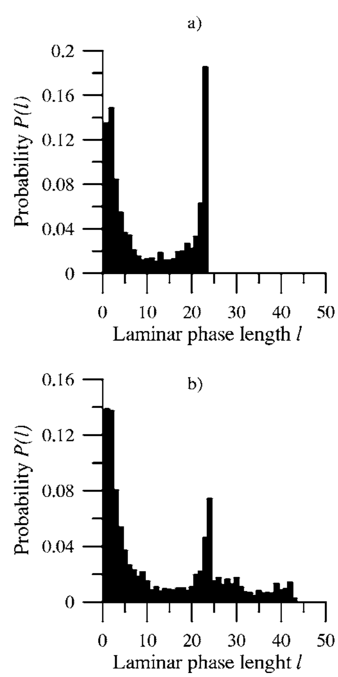

FIG. 9. A long tail in the distribution of the lengths of laminar phases in the Rössler system appears as a result of a metronomic forcing of the control parameter change: (a) stationary case; (b) nonstationary case. The parameters B and C were set as before while the parameter A was increased to 0.448 39 for 4 s with a period equal to the period of the original periodic orbit (the base value of A was 0.448 37).

III. PERIODIC ORBITS IN A SYSTEM EXHIBITING POMEAU-MANNEVILLE INTERMITTENCY UNDERGOING PERIODIC PARAMETER CHANGE

Consider a one-dimensional map in a Pomeau-Manneville intermittency state. This means that there are long periods of time during which the trajectory of the system remains close to a periodic orbit stable in another control parameter range. We will denote the period of this orbit by T.

The elements of a period-T orbit are the fixed points of the T iterate of the map,

$$f^T(a;x_i) = x_i, \tag{15}$$

where a denotes the control parameter and x_i the points belonging to the orbit. However, in our system, we do not have a stable periodic orbit but intermittency so

$$f^T(a;x_i) - x_i \neq 0. \tag{16}$$

One can then ask whether there is a value a' of the control parameter such that, if we set $a=a'$ once every T iterations, we will observe a periodic orbit. If the system returns after T iterations to the same point, it will return infinitely many times. So we can write the condition as follows:

$$f(a';f^{T-1}(a;x_i)) = x_i. \tag{17}$$

As we are interested in the laminar phase one can assume that

$$|\Delta a| \ll a, \quad a' - a = \Delta a. \tag{18}$$

In such a case, one may expand the left side of Eq. (6) into a series and neglect all nonlinear terms:

$$f(a+\Delta a;f^{T-1}(a;x_i)) = f^T(a;x_i) + \frac{\partial f(a;f^{T-1}(a;x_i))}{\partial a}\Delta a. \tag{19}$$

Using the approximation

$$f^{T-1}(a;x_i) \cong x_{i-1} \tag{20}$$

we can rewrite Eq. (17) as

$$f^T(a;x_i) + \frac{\partial f(a;x_{i-1})}{\partial a}\Delta a = x_i \tag{21}$$

and the critical value of parameter change Δa_c at which a periodic orbit appears is

$$\Delta a_C = \frac{x_i - f^T(a;x_i)}{\dfrac{\partial f(a;x_{i-1})}{\partial a}}. \tag{22}$$

This result is a special case of Eq. (9a). We can obtain Eq. (22), if we demand that the value under the square root in Eq. (9a) be infinite or negative. This leads to a nonreal value of the position of the right peak and should be interpreted as a condition for the appearance of a periodic orbit. When we set $n=T-1$, $m=1$ and set the denominator of Eq. (9a) equal to zero, we again obtain Eq. (22). These are precisely the conditions discussed in Ref. [4] at which a periodic orbit was obtained within the parameter range for stationary intermittency.

It should be stressed that the required value of the control parameter change can be either positive or negative, according to the sign of the difference in the numerator of Eq. (22) and of the derivative. In particular, the periodic orbit may appear also when one departs from the critical value of the control parameter given by Eq. (23). This is a very interesting circumstance. We expect that this property may be very useful for the control of intermittency in systems in which the critical value of the parameter cannot be obtained experimentally.

Of course, there is not only a single value of the control parameter change at which the periodic orbit appears. Such an orbit can be stabilized also if we use a value $\Delta a > \Delta a_C$ (if Δa_C is positive) or $\Delta a < \Delta a_C$ (if it is negative). In such cases the orbit moves toward the point where the modulus of Eq. (16) is greater than the smallest vertical distance between the diagonal and the T iterate of the map. It can be shown that, in most systems, especially for all systems exhibiting type I intermittency, there are two points that have this property. At one of them the periodic orbit can be stabilized.

The above described method of control of intermittency can also be applied to flows. This is possibly because a one-dimensional map may be extracted from many such flows by a variety of methods in the literature (see, e.g., [2,17]). One may then apply to these maps the method of control developed above.

Finally, we note that periodic change of the control parameter can be considered in fact to be a replacement of the original dynamical system by a new system, close to the previous one. In fact, if the value of the control parameter changes, e.g., once every T iterations, we can define a new map in the form

$$F(x) \equiv f(a';f^{T-1}(a;x)).$$

Such a map may exhibit some properties different from those of the original, simple T-iterated map, e.g., Eq. (17) represents the condition for the map F to possess a stable point. In the same way, the results of Sec. II can be rewritten in terms of the function F.

A. Example of a one-dimensional discrete system: The logistic map

The appearance of periodic orbits in a nonstationary form of the system described by the logistic equation was observed by the authors of Ref. [4]. However, there, no specific value of parameter change or condition leading to appearance of periodic orbits in this system was given. Finding the required value of parameter change is the aim of this section.

Let us set the value of the control parameter at $a=3.828$ [4]. Now we can simply compute the value of the parameter change required from Eq. (22) for all of the points forming the orbit. We then obtain three values for the parameter change for $i=1,2,3$. Two of these are negative and one is positive: $\Delta a_1 = -7.968\ldots \times 10^{-3}$; $\Delta a_2 = -6.446\ldots \times 10^{-3}$; $\Delta a_3 = 3.816\ldots \times 10^{-4}$.

Negative values of the required control parameter change obtained analytically indicate that it is possible to obtain a stable periodic orbit by moving away from the saddle-node bifurcation. One can check numerically that such an orbit appears if the value of control parameter is decreased down to 3.82 once every three iterations.

The analytical results explain why a periodic orbit was obtained in the numerical experiment of Ref. [4]. There the magnitude of the control parameter change was $\Delta a = 4.271\,225 \times 10^{-4}$ which is less than Δa_3. We now know, having computed the required values of Δa, that such an orbit can be observed also for a smaller value of the parameter change. Increasing a to 3.828 381 4 once every third

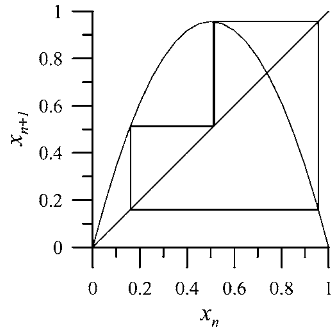

FIG. 10. Period-3 orbit obtained during type I intermittency in the logistic map as a result of periodic changes of the control parameter: the base value was 3.828 and the parameter was changed to 3.828 381 4 every three iterations.

iteration should be enough to observe a stable period-3 orbit. Figure 10 depicts this orbit.

It is also possible to observe a periodic orbit by changing the parameter once every six iterations (i.e., one iteration per two periods of the orbit). The orbit then obtained has a period of 6, but it is very similar to the period-3 orbit described above. The true structure of this orbit may be seen only in magnification. Such an orbit is shown in Fig. 11.

B. Example: The Rössler system (a flow)

We will apply our theory to the Rössler system of Sec. II B. Let us set the values of the parameters $B=2.0$ and $C=4.0$. A will be our control parameter again. As we have mentioned, the Rössler system goes through a saddle-node bifurcation at $A=0.458$. If the value of A is slightly smaller than that, type I intermittency is obtained.

To apply our theory developed for one-dimensional maps to this system one should extract such a map. We formed the first return map of the successive maxima of Y. Figure 12 depicts a plot of this function. The third iteration of this function is shown in Fig. 13 (dots) together with the polynomial fit (solid line). We can clearly see the characteristic three narrow intermittency channels.

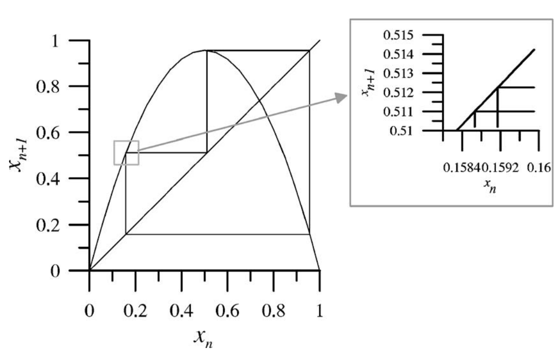

FIG. 11. Periodic orbit obtained during type I intermittency in the logistic map as a result of periodic changes of the control parameter: the base value was 3.828 and the parameter was changed to 3.828 35 every six iterations. The orbit is in fact a period-6 orbit (see magnification).

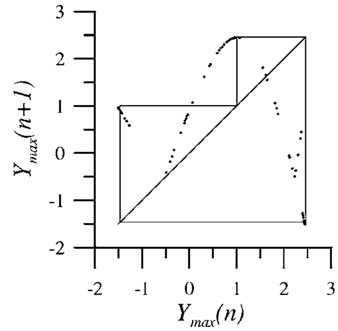

FIG. 12. First return map of the successive maxima of the variable Y of the Rössler system [Eq. (13)]. The period-3 orbit, appearing as a result of the saddle-node bifurcation which occurs for a value of the control parameter A close to that applied, is also shown.

To predict the boundary value of the parameter change at which the stable orbit will appear, it is necessary to find the value of the first derivative of the fitted function (now treated as a one-dimensional map). Computing this value is not trivial because one does not know the explicit form of the dependence of this function on the parameter A. To solve this problem, we plotted the fitting function in the vicinity of one point belonging to the periodic orbit for $A < A_C$. The slope of a linear fit to this plot yields the derivative. We then repeated this procedure for the rest of the points belonging to the periodic orbit.

If we fix the value of A at 0.4576, we can compute the differences given by Eq. (22) at all points of the orbit. At the smallest of these differences (predicted in our case to be $\Delta A=1.21\times10^{-2}$) the periodic orbit will be obtained if the control parameter is increased by ΔA for the time interval during which the trajectory moves from one maximum to the next, every three such intervals. In fact, to obtain the periodic orbit, we should change the control parameter by at least about $\Delta A=2.4\times10^{-2}$. Note that the necessary step of extracting the discrete map from the flow introduces additional errors and that our theory does not take this into account. Consequently, the numerically obtained required value of the parameter change is about twice larger than that predicted by the theory. Thus, even in the case of a flow, after a short transient, a stable periodic orbit will appear as an effect of

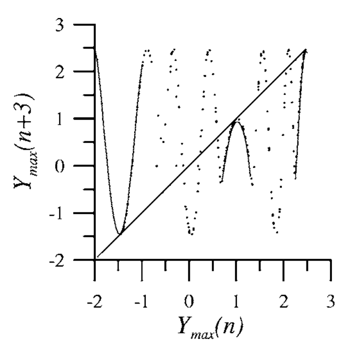

FIG. 13. Third iterate of the map of the successive maxima of the variable Y of the Rössler system [Eq. (13)] in Fig. 3 (dots). The polynomial fit is marked by the solid line. The intermittency channels are well visible.

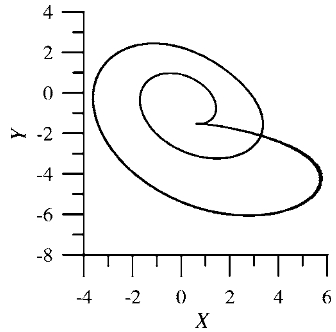

FIG. 14. Periodic orbit obtained during type I intermittency in the Rössler system by the control algorithm described in the text.

periodic parameter change provided this change is large enough (Fig. 14).

C. The periodic parameter change as a method of chaos control

Because, in a system exhibiting Pomeau-Manneville intermittency, a periodic orbit may appear due to a periodic parameter change and not due to a tangent bifurcation, we can consider this as a method of chaos control suitable for such systems. The method can be compared to orbit tracking [11–13]. Where the latter can stabilize an unstable orbit beyond a period-doubling bifurcation, our method forces the system to stay close to an orbit that would appear at a saddle-node bifurcation point at another control parameter value. The main difference is that, in our case, in the parameter range where our method is applied no periodic orbit (even an unstable one) exists. This orbit has disappeared as an effect of a tangent bifurcation. Another difference compared to all other control methods based on parameter change is that one does not need to monitor the state of the system because the parameter changes are periodic. This should simplify controlling the dynamics of a system that exhibits intermittency.

The method of control presented here requires the trajectory of the uncontrolled system to return after a *known* period of time into the vicinity of its starting point. So one can suppose that our method is appropriate for all the systems which possess this property, not only for systems exhibiting type I and type III intermittency. In fact, we were able to stabilize a periodic orbit in a system that was type V intermittent [18,19] using this method. Also, in systems exhibiting quasiperiodic motion, we managed to force the system onto a periodic orbit for a very long, albeit limited, period of time.

Although the proposed method of controlling a chaotic system has its advantages, we should stress that this algorithm can be applied only to systems completely or nearly completely free of noise. This is because the modified system is only marginally stable—the partial derivative

$$\left.\frac{\partial f^T(x)}{\partial x}\right|_{x=x_i} \cong 1.$$

For the system to be stable this derivative would have to be less than 1. The periodic change of the control parameter does not alter this. As an example, let us consider a logistic map with the base value of control parameter equal to 3.828. Change the control parameter to 3.828 427 122 5 every three iterations and add Gaussian noise with zero mean. In such a case, the control of the system will be effective only if the magnitude of noise σ is less than 3.2×10^{-5}. For comparison, the control of the logistic map with the same value of control parameter as at the unstable saddle $x_{us}\cong 0.738\,77$ using the OGY method can be applied up to the magnitude of noise 2×10^{-3}. In spite of this, we believe that the method of periodic parameter change may be useful even for systems with noise. Even should noise prevent a full stabilization of the orbit, periodic parameter change will increase the laminar phase lengths and shorten the time of chaotic bursts.

IV. CONCLUSIONS

We presented several properties of nonstationary intermittency. In Sec. II, we proposed a theory that explains the distortions of the probability distribution of the laminar phase lengths in such cases for which the periodic parameter change does not lead to a periodic orbit. We provided equations allowing to obtain the shape of such distributions. The properties of the distributions of laminar phase lengths measured during type I intermittency in human heart rate variability [4,15] may be explained by the theory discussed here.

In Sec. III we showed how a periodic change of the control parameter can lead to the appearance of a periodic orbit in a system exhibiting intermittency, in both discrete maps and flows. We showed how this property may be used as a method to control chaos during intermittency.

The logical extension of the investigations presented here will be to consider a nonstationary Pomeau-Manneville intermittency in systems with more than one dimension. Up to now, we have found nonstationary type I intermittency in a two-dimensional map and observed some of its properties similar to the one-dimensional case. We expect to find in such systems many more interesting phenomena.

ACKNOWLEDGMENTS

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APPENDIX: DERIVATION OF EQ. (10a)

We will show how Eq. (10a) is obtained for fixed values of T, n, and m.

Let us consider the map described by the equation

$$x_{n+1} = f(a;x_n). \tag{A1}$$

We know that for a value a_C close to the base value a the saddle-node bifurcation takes place and a periodic orbit appears. Assume that this is a period-3 orbit, so $T=3$. We will consider a case in which in one iteration of this map we have the base value of the parameter a and in the next one we have $a+\Delta a$, so that we may write $n=1$ and $m=1$.

The function $G(\Delta a;x_i)$ has the form

$$G(\Delta a;x_i)=f(a;f(a+\Delta a;f(a;f(a+\Delta a;f(a;f(a+\Delta a;x_i)))))). \quad \text{(A2)}$$

We want to obtain the value of

$$\left.\frac{\partial G(\Delta a;x_i)}{\partial \Delta a}\right|_{\Delta a=0}=\frac{\partial G(0;x_i)}{\partial \Delta a} \quad \text{(A3)}$$

because this derivative should be inserted into Eq. (9a). An expansion of the function (A2) into a series yields a sum of three elements, because Δa is present three times in Eq. (A2):

$$\frac{\partial G(0;x_i)}{\partial \Delta a}=A+B+C,$$

$$A=\left.\frac{\partial f(a;x)}{\partial x}\right|_{x=u_1}\frac{\partial f(a;u_2)}{\partial a},$$

$$B=\left.\frac{\partial f(a;x)}{\partial x}\right|_{x=u_1}\left.\frac{\partial f(a+\Delta a;x)}{\partial x}\right|_{x=u_2}\left.\frac{\partial f(a;x)}{\partial x}\right|_{x=u_3}\times\frac{\partial f(a;u_4)}{\partial a},$$

$$C=\left.\frac{\partial f(a;x)}{\partial x}\right|_{x=u_1}\left.\frac{\partial f(a+\Delta a;x)}{\partial x}\right|_{x=u_2}\left.\frac{\partial f(a;x)}{\partial x}\right|_{x=u_3}\times\left.\frac{\partial f(a+\Delta a;x)}{\partial x}\right|_{x=u_4}\left.\frac{\partial f(a;x)}{\partial x}\right|_{x=u_5}\frac{\partial f(a;u_6)}{\partial a}. \quad \text{(A4)}$$

In the above, we have

$$u_1=f(a+\Delta a;f(a;f(a+\Delta a;f(a;f(a+\Delta a;x_i))))),$$

$$u_2=f(a;f(a+\Delta a;f(a;f(a+\Delta a;x_i)))),$$

$$\vdots$$

$$u_6\equiv x_i. \quad \text{(A5)}$$

If we take into account that the value of a (and $a+\Delta a$ of course, as Δa is close or equal to zero) is close to the critical value a_C at which the periodic orbit appears, we can apply the approximation

$$f(a;x_i)\cong x_{i+1} \quad \text{(A6)}$$

which leads to

$$u_6\equiv x_i,$$

$$u_5=f(a+\Delta a;x_i)\cong x_{i+1},$$

$$u_4=f(a;f(a+\Delta a;x_i))\cong f(a;x_{i+1})\cong x_{i+2},$$

$$\vdots$$

$$u_k\cong x_{i+6-k},\quad k=1,2,\ldots,6,$$

$$x_{i+3}\equiv x_i. \quad \text{(A7)}$$

Substituting it into (A4) we obtain

$$\frac{\partial G(0;x_i)}{\partial \Delta a}=A+B+C,$$

$$A=\frac{\partial f(a;x_{i+5})}{\partial x}\frac{\partial f(a;x_{i+4})}{\partial a},$$

$$B=\prod_{j=i+3}^{i+5}\frac{\partial f(a;x_j)}{\partial x}\frac{\partial f(a;x_{i+2})}{\partial a},$$

$$C=\prod_{j=i+1}^{i+5}\frac{\partial f(a;x_j)}{\partial x}\frac{\partial f(a;x_i)}{\partial a}. \quad \text{(A8)}$$

The sum (A8) can be written in the more compact form

$$\frac{\partial G(0;x_i)}{\partial \Delta a}=\sum_{q=1}^{3}\left(\prod_{k=2q}^{6}\frac{\partial f(a;x_{i+k-1})}{\partial x}\right)\frac{\partial f(a;x_{i+2q-2})}{\partial a}. \quad \text{(A9)}$$

If m were greater than 1, in Eq. (A9) one sum more would appear. It is not difficult to deduct the form of this sum and to rewrite (A9) as

$$\frac{\partial G(0;x_i)}{\partial \Delta a}=\sum_{q=1}^{3}\sum_{j=1q+1(q-1)}^{2q-1}\left(\prod_{k=j+1}^{6}\frac{\partial f(a;x_{i+k-1})}{\partial x}\right)\frac{\partial f(a;x_{i+j-1})}{\partial a}. \quad \text{(A10)}$$

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